CHAPTER 2
Limits and Continuity

EXERCISE SET 2.1

1. **(a)** -1 **(b)** 3 **(c)** does not exist
(d) 1 **(e)** -1 **(f)** 3

2. **(a)** 2 **(b)** 0 **(c)** does not exist
(d) 2 **(e)** 0 **(f)** 2

3. **(a)** 1 **(b)** 1 **(c)** 1 **(d)** 1 **(e)** $-\infty$ **(f)** $+\infty$

4. **(a)** 3 **(b)** 3 **(c)** 3 **(d)** 3 **(e)** $+\infty$ **(f)** $+\infty$

5. **(a)** 0 **(b)** 0 **(c)** 0 **(d)** 3 **(e)** $+\infty$ **(f)** $+\infty$

6. **(a)** 2 **(b)** 2 **(c)** 2 **(d)** 3 **(e)** $-\infty$ **(f)** $+\infty$

7. **(a)** $-\infty$ **(b)** $+\infty$ **(c)** does not exist
(d) undef **(e)** 2 **(f)** 0

8. **(a)** $+\infty$ **(b)** $+\infty$ **(c)** $+\infty$ **(d)** undef **(e)** 0 **(f)** -1

9. **(a)** $-\infty$ **(b)** $-\infty$ **(c)** $-\infty$ **(d)** 1 **(e)** 1 **(f)** 2

10. **(a)** 1 **(b)** $-\infty$ **(c)** does not exist
(d) -2 **(e)** $+\infty$ **(f)** $+\infty$

11. **(a)** 0 **(b)** 0 **(c)** 0
(d) 0 **(e)** does not exist **(f)** does not exist

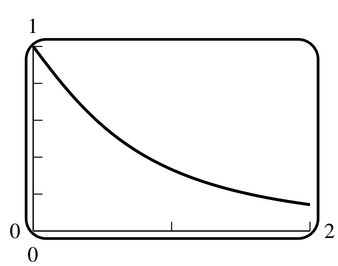
12. **(a)** 3 **(b)** 3 **(c)** 3
(d) 3 **(e)** does not exist **(f)** 0

13. for all $x_0 \neq -4$

14. for all $x_0 \neq -6, 3$

19. **(a)**

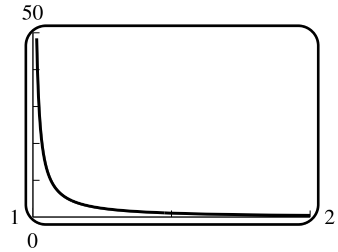
2	1.5	1.1	1.01	1.001	0	0.5	0.9	0.99	0.999
0.1429	0.2105	0.3021	0.3300	0.3330	1.0000	0.5714	0.3690	0.3367	0.3337



The limit is 1/3.

(b)

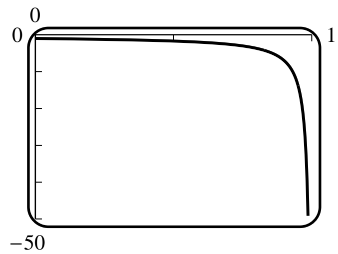
2	1.5	1.1	1.01	1.001	1.0001
0.4286	1.0526	6.344	66.33	666.3	6666.3



The limit is $+\infty$.

(c)

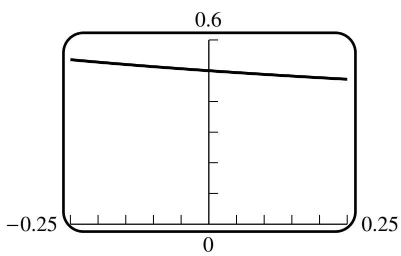
0	0.5	0.9	0.99	0.999	0.9999
-1	-1.7143	-7.0111	-67.001	-667.0	-6667.0



The limit is $-\infty$.

20. (a)

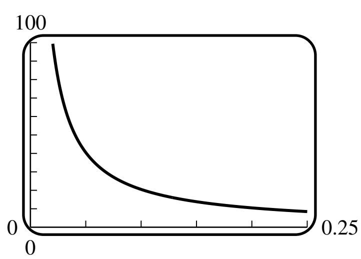
-0.25	-0.1	-0.001	-0.0001	0.0001	0.001	0.1	0.25
0.5359	0.5132	0.5001	0.5000	0.5000	0.4999	0.4881	0.4721



The limit is $1/2$.

(b)

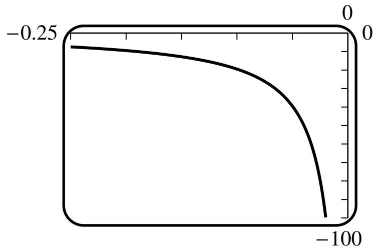
0.25	0.1	0.001	0.0001
8.4721	20.488	2000.5	20001



The limit is $+\infty$.

(c)

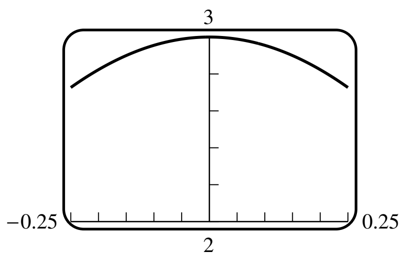
−0.25	−0.1	−0.001	−0.0001
−7.4641	−19.487	−1999.5	−20000



The limit is $-\infty$.

21. **(a)**

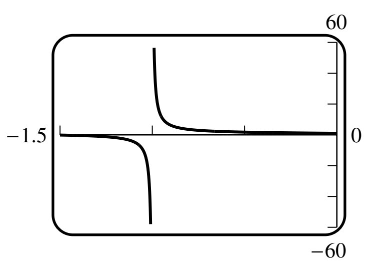
−0.25	−0.1	−0.001	−0.0001	0.0001	0.001	0.1	0.25
2.7266	2.9552	3.0000	3.0000	3.0000	3.0000	2.9552	2.7266



The limit is 3.

(b)

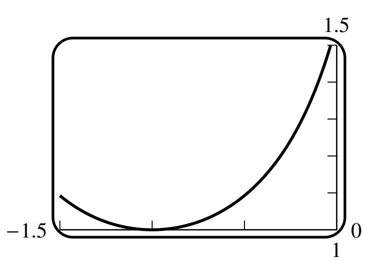
0	−0.5	−0.9	−0.99	−0.999	−1.5	−1.1	−1.01	−1.001
1	1.7552	6.2161	54.87	541.1	−0.1415	−4.536	−53.19	−539.5



The limit does not exist.

22. **(a)**

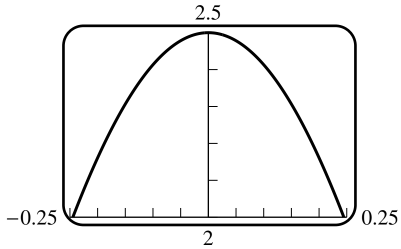
0	−0.5	−0.9	−0.99	−0.999	−1.5	−1.1	−1.01	−1.001
1.5574	1.0926	1.0033	1.0000	1.0000	1.0926	1.0033	1.0000	1.0000



The limit is 1.

(b)

−0.25	−0.1	−0.001	−0.0001	0.0001	0.001	0.1	0.25
1.9794	2.4132	2.5000	2.5000	2.5000	2.5000	2.4132	1.9794



The limit is 5/2.

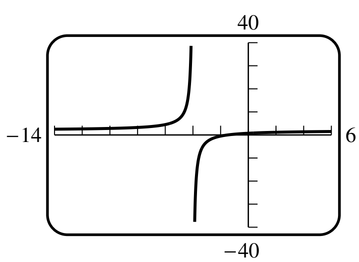
23. The height of the ball at time $t = 0.25 + \Delta t$ is $s(0.25 + \Delta t) = -16(0.25 + \Delta t)^2 + 29(0.25 + \Delta t) + 6$, so the distance traveled over the interval from $t = 0.25 - \Delta t$ to $t = 0.25 + \Delta t$ is
$s(0.25 + \Delta t) - s(0.25 - \Delta t) = -64(0.25)\Delta t + 58\Delta t$.
Thus the average velocity over the same interval is given by
$v_{\text{ave}} = [s(0.25 + \Delta t) - s(0.25 - \Delta t)]/2\Delta t = (-64(0.25)\Delta t + 58\Delta t)/2\Delta t = 21$ ft/s,
and this will also be the instantaneous velocity, since it happens to be independent of Δt.

24. The height of the ball at time $t = 0.75 + \Delta t$ is $s(0.75 + \Delta t) = -16(0.75 + \Delta t)^2 + 29(0.75 + \Delta t) + 6$, so the distance traveled over the interval from $t = 0.75 - \Delta t$ to $t = 0.75 + \Delta t$ is
$s(0.75 + \Delta t) - s(0.75 - \Delta t) = -64(0.75)\Delta t + 58\Delta t$.
Thus the average velocity over the same interval is given by
$v_{\text{ave}} = [s(0.75 + \Delta t) - s(0.75 - \Delta t)]/2\Delta t = (-64(0.75)\Delta t + 58\Delta t)/2\Delta t = 5$ ft/s,
and this will also be the instantaneous velocity, since it happens to be independent of Δt.

25. **(a)**

−100,000,000	−100,000	−1000	−100	−10	10	100	1000
2.0000	2.0001	2.0050	2.0521	2.8333	1.6429	1.9519	1.9950

100,000	100,000,000
2.0000	2.0000

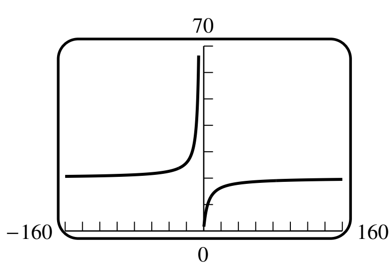


asymptote $y = 2$ as $x \to \pm\infty$

(b)

−100,000,000	−100,000	−1000	−100	−10	10	100	1000
20.0855	20.0864	20.1763	21.0294	35.4013	13.7858	19.2186	19.9955

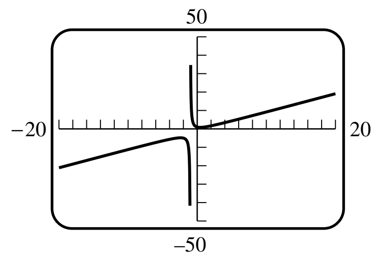
100,000	100,000,000
20.0846	20.0855



asymptote $y = 20.086$.

(c)

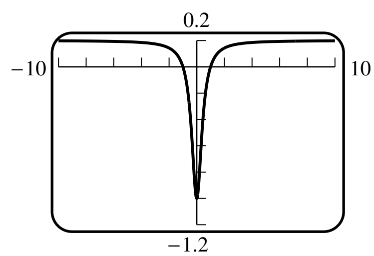
−100,000,000	−100,000	−1000	−100	−10	10	100	1000	100,000	100,000,000
−100,000,001	−100,000	−1001	−101.0	−11.2	9.2	99.0	999.0	99,999	99,999,999



no horizontal asymptote

26. **(a)**

−100,000,000	−100,000	−1000	−100	−10	10	100	1000	100,000	100,000,000
0.2000	0.2000	0.2000	0.2000	0.1976	0.1976	0.2000	0.2000	0.2000	0.2000

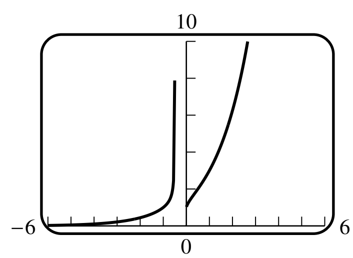


asymptote $y = 1/5$ as $x \to \pm\infty$

(b)

−100,000,000	−100,000	−1000	−100	−10	10	100
0.0000	0.0000	0.0000	0.0000	0.0016	1668.0	2.09×10^{18}

1000	100,000	100,000,000
1.77×10^{301}	?	?

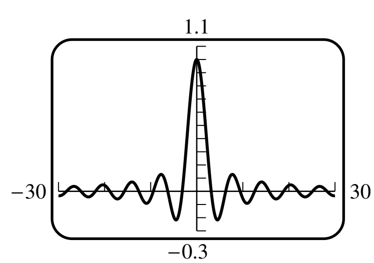


asymptote $y = 0$ as $x \to -\infty$, none as $x \to +\infty$

(c)

−100,000,000	−100,000	−1000	−100	−10	10	100
0.0000	0.0000	0.0008	−0.0051	−0.0544	−0.0544	−0.0051

1000	100,000	100,000,000
0.0008	0.0000	0.0000



asymptote $y = 0$ as $x \to \pm\infty$

27. It appears that $\lim_{t\to+\infty} n(t) = +\infty$, and $\lim_{t\to+\infty} e(t) = c$.

28. **(a)** It is the initial temperature of the oven.
(b) It is the ambient temperature, i.e. the temperature of the room.

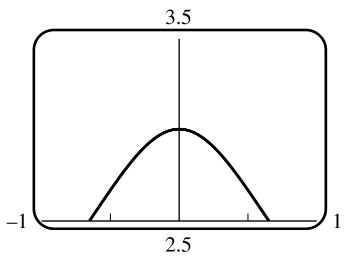
29. **(a)** $\lim_{t\to 0^+} \frac{\sin t}{t}$ **(b)** $\lim_{t\to 0^+} \frac{t-1}{t+1}$ **(c)** $\lim_{t\to 0^-} (1+2t)^{1/t}$

30. **(a)** $\lim_{t\to 0^+} \frac{\cos \pi t}{\pi t}$ **(b)** $\lim_{t\to 0^+} \frac{1}{t+1}$ **(c)** $\lim_{t\to 0^-} \left(1+\frac{2}{t}\right)^t$

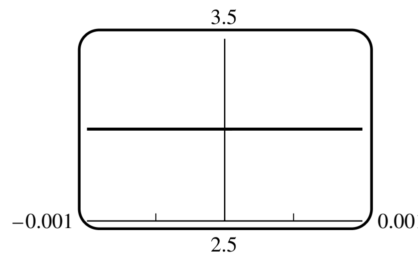
31. $\lim_{x\to-\infty} f(x) = L$ and $\lim_{x\to+\infty} = L$

32. **(a)** no
(b) yes; $\tan x$ and $\sec x$ at $x = n\pi + \pi/2$, and $\cot x$ and $\csc x$ at $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

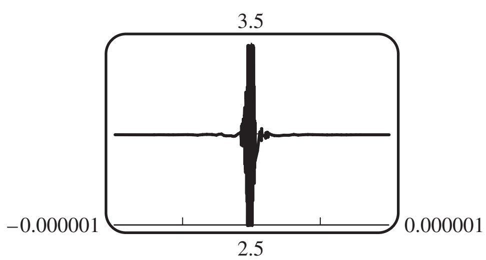
33. **(a)** The limit appears to be 3.



(b) The limit appears to be 3.



(c) The limit does not exist.



35. **(a)** The plot over the interval $[-a, a]$ becomes subject to catastrophic subtraction if a is small enough (the size depending on the machine).
(c) It does not.

EXERCISE SET 2.2

1. **(a)** 7 **(b)** π **(c)** -6 **(d)** 36

2. **(a)** 1 **(b)** -1 **(c)** 1 **(d)** -1

3. **(a)** -6 **(b)** 13 **(c)** -8 **(d)** 16 **(e)** 2 **(f)** $-1/2$
(g) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.
(h) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.

4. **(a)** 0
(b) The limit doesn't exist because $\lim f$ doesn't exist and $\lim g$ does.
(c) 0 **(d)** 3 **(e)** 0
(f) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.
(g) The limit doesn't exist because $\sqrt{f(x)}$ is not defined for $0 \le x < 2$.
(h) 1

5. 0 **6.** 3/4 **7.** 8 **8.** -3

9. 4 **10.** 12 **11.** $-4/5$ **12.** 0

13. 3/2 **14.** 4/3 **15.** $+\infty$ **16.** $-\infty$

17. does not exist **18.** $+\infty$ **19.** $-\infty$ **20.** does not exist

21. $+\infty$ **22.** $-\infty$ **23.** does not exist **24.** $-\infty$

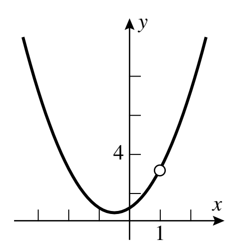
25. $+\infty$ **26.** does not exist **27.** $+\infty$ **28.** $+\infty$

29. 6 **30.** 4 **32.** -19

33. **(a)** 2 **(b)** 2 **(c)** 2

34. **(a)** -2 **(b)** 0 **(c)** does not exist

35. **(a)** 3 **(b)**



36. **(a)** -6 **(b)** $F(x) = x - 3$

37. **(a)** Theorem 2.2.2(a) doesn't apply; moreover one cannot add/subtract infinities.
(b) $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right) = \lim\limits_{x\to 0^+}\left(\dfrac{x-1}{x^2}\right) = -\infty$

38. $\lim\limits_{x\to 0^-}\left(\dfrac{1}{x} + \dfrac{1}{x^2}\right) = \lim\limits_{x\to 0^-}\dfrac{x+1}{x^2} = +\infty$ **39.** $\lim\limits_{x\to 0}\dfrac{x}{x\left(\sqrt{x+4}+2\right)} = \dfrac{1}{4}$

40. $\lim\limits_{x\to 0}\dfrac{x^2}{x\left(\sqrt{x+4}+2\right)} = 0$

41. The left and/or right limits could be plus or minus infinity; or the limit could exist, or equal any preassigned real number. For example, let $q(x) = x - x_0$ and let $p(x) = a(x - x_0)^n$ where n takes on the values $0, 1, 2$.

EXERCISE SET 2.3

1. **(a)** -3 **(b)** $-\infty$

2. **(a)** 1 **(b)** -1

3. **(a)** -12 **(b)** 21 **(c)** -15 **(d)** 25
(e) 2 **(f)** $-3/5$ **(g)** 0
(h) The limit doesn't exist because the denominator tends to zero but the numerator doesn't.

4. **(a)** 20 **(b)** 0 **(c)** $+\infty$ **(d)** $-\infty$
(e) $-42^{1/3}$ **(f)** $-6/7$ **(g)** 7 **(h)** $-7/12$

5. $+\infty$ **6.** 5 **7.** $-\infty$ **8.** $+\infty$

9. $+\infty$ **10.** $+\infty$ **11.** $3/2$ **12.** $5/2$

13. 0 **14.** 0 **15.** 0 **16.** $5/3$

17. $-5^{1/3}/2$ **18.** $\sqrt[3]{3/2}$ **19.** $-\sqrt{5}$ **20.** $\sqrt{5}$

21. $1/\sqrt{6}$ **22.** $-1/\sqrt{6}$ **23.** $\sqrt{3}$ **24.** $\sqrt{3}$

25. $-\infty$ **26.** $+\infty$ **27.** $-1/7$ **28.** $4/7$

29. **(a)** $+\infty$ **(b)** -5

30. **(a)** 0 **(b)** -6

31. $\displaystyle\lim_{x\to+\infty}(\sqrt{x^2+3}-x)\frac{\sqrt{x^2+3}+x}{\sqrt{x^2+3}+x}=\lim_{x\to+\infty}\frac{3}{\sqrt{x^2+3}+x}=0$

32. $\displaystyle\lim_{x\to+\infty}(\sqrt{x^2-3x}-x)\frac{\sqrt{x^2-3x}+x}{\sqrt{x^2-3x}+x}=\lim_{x\to+\infty}\frac{-3x}{\sqrt{x^2-3x}+x}=-3/2$

33. $\displaystyle\lim_{x\to+\infty}\left(\sqrt{x^2+ax}-x\right)\frac{\sqrt{x^2+ax}+x}{\sqrt{x^2+ax}+x}=\lim_{x\to+\infty}\frac{ax}{\sqrt{x^2+ax}+x}=a/2$

34. $\displaystyle\lim_{x\to+\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\frac{\sqrt{x^2+ax}+\sqrt{x^2+bx}}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to+\infty}\frac{(a-b)x}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\frac{a-b}{2}$

35. $\displaystyle\lim_{x\to+\infty}p(x)=(-1)^n\infty$ and $\displaystyle\lim_{x\to-\infty}p(x)=+\infty$

36. If $m>n$ the limits are both zero. If $m=n$ the limits are both 1. If $n>m$ the limits are $(-1)^{n+m}\infty$ and $+\infty$, respectively.

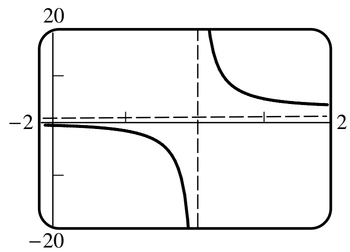
37. If $m>n$ the limits are both zero. If $m=n$ the limits are both equal to a_m, the leading coefficient of p. If $n>m$ the limits are $\pm\infty$ where the sign depends on the sign of a_m and whether n is even or odd.

38. **(a)** $p(x)=q(x)=x$ **(b)** $p(x)=x$, $q(x)=x^2$
(c) $p(x)=x^2$, $q(x)=x$ **(d)** $p(x)=x+3$, $q(x)=x$

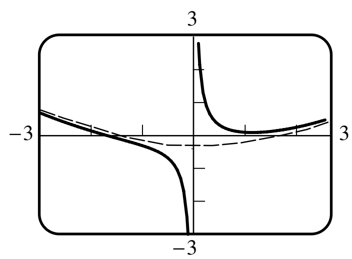
39. If $m > n$ the limit is 0. If $m = n$ the limit is -3. If $m < n$ and $n - m$ is odd, then the limit is $+\infty$; if $m < n$ and $n - m$ is even, then the limit is $-\infty$.

40. If $m > n$ the limit is zero. If $m = n$ the limit is c_m/d_m. If $n > m$ the limit is $\pm\infty$, where the sign depends on the signs of c_n and d_m.

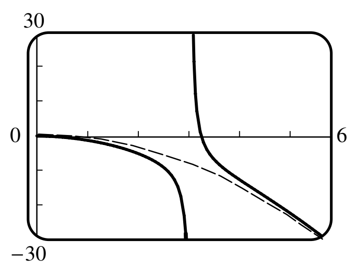
41. $f(x) = x + 2 + \dfrac{2}{x-2}$, so $\lim_{x\to\pm\infty}(f(x) - (x+2)) = 0$ and $f(x)$ is asymptotic to $y = x + 2$.



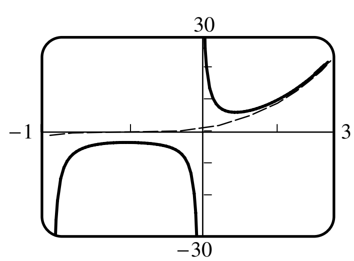
42. $f(x) = x^2 - 1 + 3/x$, so $\lim_{x\to\pm\infty}[f(x) - (x^2 - 1) = 0]$ and $f(x)$ is asymptotic to $y = x^2 - 1$.



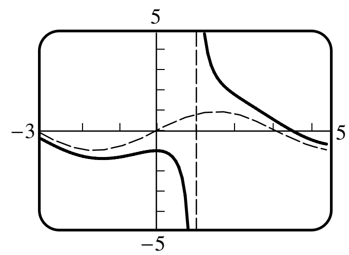
43. $f(x) = -x^2 + 1 + 2/(x-3)$ so $\lim_{x\to\pm\infty}[f(x) - (-x^2 + 1)] = 0$ and $f(x)$ is asymptotic to $y = -x^2 + 1$.



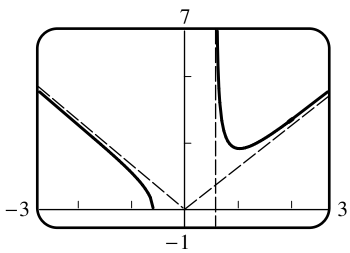
44. $f(x) = x^3 + \dfrac{3}{2(x-1)} - \dfrac{3}{2(x+1)}$ so $\lim_{x\to\pm\infty}[f(x) - x^3] = 0$ and $f(x)$ is asymptotic to $y = x^3$.



45. $f(x) - \sin x = 0$ and $f(x)$ is asymptotic to $y = \sin x$.



46. Note that the function is not defined for $-1 < x <= 1$. For x outside this interval we have $f(x) = \sqrt{x^2 + \dfrac{2}{x-1}}$ which suggests that $\lim_{x\to\pm\infty}[f(x) - |x|\,] = 0$ (this can be checked with a CAS) and hence $f(x)$ is asymptotic to $y = |x|$.



EXERCISE SET 2.4

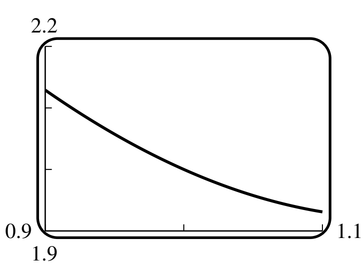
1. **(a)** $|f(x) - f(0)| = |x + 2 - 2| = |x| < 0.1$ if and only if $|x| < 0.1$
(b) $|f(x) - f(3)| = |(4x - 5) - 7| = 4|x - 3| < 0.1$ if and only if $|x - 3| < (0.1)/4 = 0.0025$
(c) $|f(x) - f(4)| = |x^2 - 16| < \epsilon$ if $|x - 4| < \delta$. We get $f(x) = 16 + \epsilon = 16.001$ at $x = 4.000124998$, which corresponds to $\delta = 0.000124998$; and $f(x) = 16 - \epsilon = 15.999$ at $x = 3.999874998$, for which $\delta = 0.000125002$. Use the smaller δ: thus $|f(x) - 16| < \epsilon$ provided $|x - 4| < 0.000125$ (to six decimals).

2. **(a)** $|f(x) - f(0)| = |2x + 3 - 3| = 2|x| < 0.1$ if and only if $|x| < 0.05$
(b) $|f(x) - f(0)| = |2x + 3 - 3| = 2|x| < 0.01$ if and only if $|x| < 0.005$
(c) $|f(x) - f(0)| = |2x + 3 - 3| = 2|x| < 0.0012$ if and only if $|x| < 0.0006$

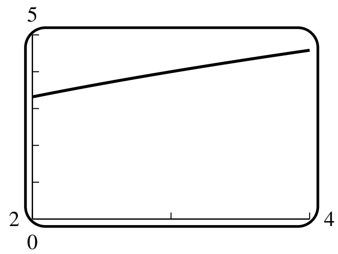
3. **(a)** $x_1 = (1.95)^2 = 3.8025, x_2 = (2.05)^2 = 4.2025$
(b) $\delta = \min(\,|4 - 3.8025|, |4 - 4.2025|\,) = 0.1975$

4. **(a)** $x_1 = 1/(1.1) = 0.909090\ldots, x_2 = 1/(0.9) = 1.111111\ldots$
(b) $\delta = \min(\,|1 - 0.909090|, |1 - 1.111111|\,) = 0.0909090\ldots$

5. $|(x^3-4x+5)-2| < 0.05$, $-0.05 < (x^3-4x+5)-2 < 0.05$, $1.95 < x^3-4x+5 < 2.05$; $x^3-4x+5 = 1.95$ at $x = 1.0616$, $x^3 - 4x + 5 = 2.05$ at $x = 0.9558$; $\delta = \min(1.0616 - 1, 1 - 0.9558) = 0.0442$



6. $\sqrt{5x+1} = 3.5$ at $x = 2.25$, $\sqrt{5x+1} = 4.5$ at $x = 3.85$, so $\delta = \min(3 - 2.25, 3.85 - 3) = 0.75$



7. With the TRACE feature of a calculator we discover that (to five decimal places) $(0.87000, 1.80274)$ and $(1.13000, 2.19301)$ belong to the graph. Set $x_0 = 0.87$ and $x_1 = 1.13$. Since the graph of $f(x)$ rises from left to right, we see that if $x_0 < x < x_1$ then $1.80274 < f(x) < 2.19301$, and therefore $1.8 < f(x) < 2.2$. So we can take $\delta = 0.13$.

8. From a calculator plot we conjecture that $\lim_{x\to 1} f(x) = 2$. Using the TRACE feature we see that the points $(\pm 0.2, 1.94709)$ belong to the graph. Thus if $-0.2 < x < 0.2$ then $1.95 < f(x) \le 2$ and hence $|f(x) - L| < 0.05 < 0.1 = \epsilon$.

9. $|2x - 8| = 2|x - 4| < 0.1$ if $|x - 4| < 0.05$, $\delta = 0.05$

10. $|x/2 + 1| = (1/2)|x - (-2)| < 0.1$ if $|x + 2| < 0.2$, $\delta = 0.2$

11. $|7x + 5 - (-2)| = 7|x - (-1)| < 0.01$ if $|x + 1| < \frac{1}{700}$, $\delta = \frac{1}{700}$

12. $|5x - 2 - 13| = 5|x - 3| < 0.01$ if $|x - 3| < \frac{1}{500}$, $\delta = \frac{1}{500}$

13. $\left|\dfrac{x^2-4}{x-2} - 4\right| = \left|\dfrac{x^2 - 4 - 4x + 8}{x-2}\right| = |x - 2| < 0.05$ if $|x - 2| < 0.05$, $\delta = 0.05$

14. $\left|\dfrac{x^2-1}{x+1} - (-2)\right| = \left|\dfrac{x^2 - 1 + 2x + 2}{x+1}\right| = |x + 1| < 0.05$ if $|x + 1| < 0.05$, $\delta = 0.05$

15. if $\delta < 1$ then $\left|x^2 - 16\right| = |x - 4||x + 4| < 9|x - 4| < 0.001$ if $|x - 4| < \frac{1}{9000}$, $\delta = \frac{1}{9000}$

16. if $\delta < 1$ then $|\sqrt{x} - 3|\left|\dfrac{\sqrt{x}+3}{\sqrt{x}+3}\right| = \dfrac{|x-9|}{|\sqrt{x}+3|} < \dfrac{|x-9|}{\sqrt{8}+3} < \dfrac{1}{4}|x - 9| < 0.001$ if $|x - 9| < 0.004$, $\delta = 0.004$

17. if $\delta \le 1$ then $\left|\dfrac{1}{x} - \dfrac{1}{5}\right| = \dfrac{|x-5|}{5|x|} \le \dfrac{|x-5|}{20} < 0.05$ if $|x - 5| < 1$, $\delta = 1$

18. $|x - 0| = |x| < 0.05$ if $|x| < 0.05$, $\delta = 0.05$

19. $|3x - 15| = 3|x - 5| < \epsilon$ if $|x - 5| < \frac{1}{3}\epsilon$, $\delta = \frac{1}{3}\epsilon$

20. $|(4x - 5) - 7| = |4x - 12| = 4|x - 3| < \epsilon$ if $|x - 3| < \frac{1}{4}\epsilon$, $\delta = \frac{1}{4}\epsilon$

21. $|2x - 7 - (-3)| = 2|x - 2| < \epsilon$ if $|x - 2| < \frac{1}{2}\epsilon$, $\delta = \frac{1}{2}\epsilon$

22. $|2 - 3x - 5| = 3|x + 1| < \epsilon$ if $|x + 1| < \frac{1}{3}\epsilon$, $\delta = \frac{1}{3}\epsilon$

23. $\left|\dfrac{x^2+x}{x} - 1\right| = |x| < \epsilon$ if $|x| < \epsilon$, $\delta = \epsilon$

24. $\left|\dfrac{x^2-9}{x+3}-(-6)\right| = |x+3| < \epsilon$ if $|x+3| < \epsilon$, $\delta = \epsilon$

25. if $\delta < 1$ then $|2x^2-2| = 2|x-1||x+1| < 6|x-1| < \epsilon$ if $|x-1| < \frac{1}{6}\epsilon$, $\delta = \min(1, \frac{1}{6}\epsilon)$

26. if $\delta < 1$ then $|x^2-5-4| = |x-3||x+3| < 7|x-3| < \epsilon$ if $|x-3| < \frac{1}{7}\epsilon$, $\delta = \min(1, \frac{1}{7}\epsilon)$

27. if $\delta < \dfrac{1}{6}$ then $\left|\dfrac{1}{x}-3\right| = \dfrac{3|x-\frac{1}{3}|}{|x|} < 18\left|x-\dfrac{1}{3}\right| < \epsilon$ if $\left|x-\dfrac{1}{3}\right| < \frac{1}{18}\epsilon$, $\delta = \min\left(\dfrac{1}{6}, \dfrac{1}{18}\epsilon\right)$

28. If $\delta < \frac{1}{2}$ and $|x-(-2)| < \delta$ then $-\frac{5}{2} < x < -\frac{3}{2}$, $x+1 < -\frac{1}{2}$, $|x+1| > \frac{1}{2}$; then $\left|\dfrac{1}{x+1}-(-1)\right| = \dfrac{|x+2|}{|x+1|} < 2|x+2| < \epsilon$ if $|x+2| < \dfrac{1}{2}\epsilon$, $\delta = \min\left(\dfrac{1}{2}, \dfrac{1}{2}\epsilon\right)$

29. $|\sqrt{x}-2| = \left|(\sqrt{x}-2)\dfrac{\sqrt{x}+2}{\sqrt{x}+2}\right| = \left|\dfrac{x-4}{\sqrt{x}+2}\right| < \dfrac{1}{2}|x-4| < \epsilon$ if $|x-4| < 2\epsilon$, $\delta = 2\epsilon$

30. $|\sqrt{x+3}-3|\left|\dfrac{\sqrt{x+3}+3}{\sqrt{x+3}+3}\right| = \dfrac{|x-6|}{\sqrt{x+3}+3} \le \dfrac{1}{3}|x-6| < \epsilon$ if $|x-6| < 3\epsilon$, $\delta = 3\epsilon$

31. $|f(x)-3| = |x+2-3| = |x-1| < \epsilon$ if $0 < |x-1| < \epsilon$, $\delta = \epsilon$

32. If $\delta < 1$ then $|(x^2+3x-1)-9| = |(x-2)(x+5)| < 8|x-2| < \epsilon$ if $|x-2| < \frac{1}{8}\epsilon$, $\delta = \min\left(1, \frac{1}{8}\epsilon\right)$

33. **(a)** $|f(x)-L| = \dfrac{1}{x^2} < 0.1$ if $x > \sqrt{10}$, $N = \sqrt{10}$

(b) $|f(x)-L| = \left|\dfrac{x}{x+1}-1\right| = \left|\dfrac{1}{x+1}\right| < 0.01$ if $x+1 > 100$, $N = 99$

(c) $|f(x)-L| = \left|\dfrac{1}{x^3}\right| < \dfrac{1}{1000}$ if $|x| > 10$, $x < -10$, $N = -10$

(d) $|f(x)-L| = \left|\dfrac{x}{x+1}-1\right| = \left|\dfrac{1}{x+1}\right| < 0.01$ if $|x+1| > 100$, $-x-1 > 100$, $x < -101$, $N = -101$

34. **(a)** $\left|\dfrac{1}{x^3}\right| < 0.1$, $x > 10^{1/3}$, $N = 10^{1/3}$ **(b)** $\left|\dfrac{1}{x^3}\right| < 0.01$, $x > 100^{1/3}$, $N = 100^{1/3}$

(c) $\left|\dfrac{1}{x^3}\right| < 0.001$, $x > 10$, $N = 10$

35. **(a)** $\dfrac{x_1^2}{1+x_1^2} = 1-\epsilon$, $x_1 = -\sqrt{\dfrac{1-\epsilon}{\epsilon}}$; $\dfrac{x_2^2}{1+x_2^2} = 1-\epsilon$, $x_2 = \sqrt{\dfrac{1-\epsilon}{\epsilon}}$

(b) $N = \sqrt{\dfrac{1-\epsilon}{\epsilon}}$ **(c)** $N = -\sqrt{\dfrac{1-\epsilon}{\epsilon}}$

36. **(a)** $x_1 = -1/\epsilon^3$; $x_2 = 1/\epsilon^3$ **(b)** $N = 1/\epsilon^3$ **(c)** $N = -1/\epsilon^3$

37. $\dfrac{1}{x^2} < 0.01$ if $|x| > 10$, $N = 10$

38. $\dfrac{1}{x+2} < 0.005$ if $|x+2| > 200$, $x > 198$, $N = 198$

39. $\left|\dfrac{x}{x+1}-1\right| = \left|\dfrac{1}{x+1}\right| < 0.001$ if $|x+1| > 1000$, $x > 999$, $N = 999$

40. $\left|\dfrac{4x-1}{2x+5}-2\right| = \left|\dfrac{11}{2x+5}\right| < 0.1$ if $|2x+5| > 110$, $2x > 105$, $N = 52.5$

41. $\left|\dfrac{1}{x+2}-0\right| < 0.005$ if $|x+2| > 200$, $-x-2 > 200$, $x < -202$, $N = -202$

42. $\left|\dfrac{1}{x^2}\right| < 0.01$ if $|x| > 10$, $-x > 10$, $x < -10$, $N = -10$

43. $\left|\dfrac{4x-1}{2x+5}-2\right| = \left|\dfrac{11}{2x+5}\right| < 0.1$ if $|2x+5| > 110$, $-2x-5 > 110$, $2x < -115$, $x < -57.5$, $N = -57.5$

44. $\left|\dfrac{x}{x+1}-1\right| = \left|\dfrac{1}{x+1}\right| < 0.001$ if $|x+1| > 1000$, $-x-1 > 1000$, $x < -1001$, $N = -1001$

45. $\left|\dfrac{1}{x^2}\right| < \epsilon$ if $|x| > \dfrac{1}{\sqrt{\epsilon}}$, $N = \dfrac{1}{\sqrt{\epsilon}}$

46. $\left|\dfrac{1}{x}\right| < \epsilon$ if $|x| > \dfrac{1}{\epsilon}$, $-x > \dfrac{1}{\epsilon}$, $x < -\dfrac{1}{\epsilon}$, $N = -\dfrac{1}{\epsilon}$

47. $\left|\dfrac{1}{x+2}\right| < \epsilon$ if $|x+2| > \dfrac{1}{\epsilon}$, $-x-2 < \dfrac{1}{\epsilon}$, $x > -2-\dfrac{1}{\epsilon}$, $N = -2-\dfrac{1}{\epsilon}$

48. $\left|\dfrac{1}{x+2}\right| < \epsilon$ if $|x+2| > \dfrac{1}{\epsilon}$, $x+2 > \dfrac{1}{\epsilon}$, $x > \dfrac{1}{\epsilon}-2$, $N = \dfrac{1}{\epsilon}-2$

49. $\left|\dfrac{x}{x+1}-1\right| = \left|\dfrac{1}{x+1}\right| < \epsilon$ if $|x+1| > \dfrac{1}{\epsilon}$, $x > \dfrac{1}{\epsilon}-1$, $N = \dfrac{1}{\epsilon}-1$

50. $\left|\dfrac{x}{x+1}-1\right| = \left|\dfrac{1}{x+1}\right| < \epsilon$ if $|x+1| > \dfrac{1}{\epsilon}$, $-x-1 > \dfrac{1}{\epsilon}$, $x < -1-\dfrac{1}{\epsilon}$, $N = -1-\dfrac{1}{\epsilon}$

51. $\left|\dfrac{4x-1}{2x+5}-2\right| = \left|\dfrac{11}{2x+5}\right| < \epsilon$ if $|2x+5| > \dfrac{11}{\epsilon}$, $-2x-5 > \dfrac{11}{\epsilon}$, $2x < -\dfrac{11}{\epsilon}-5$, $x < -\dfrac{11}{2\epsilon}-\dfrac{5}{2}$,
$N = -\dfrac{5}{2}-\dfrac{11}{2\epsilon}$

52. $\left|\dfrac{4x-1}{2x+5}-2\right| = \left|\dfrac{11}{2x+5}\right| < \epsilon$ if $|2x+5| > \dfrac{11}{\epsilon}$, $2x > \dfrac{11}{\epsilon}-5$, $x > \dfrac{11}{2\epsilon}-\dfrac{5}{2}$, $N = \dfrac{11}{2\epsilon}-\dfrac{5}{2}$

53. **(a)** $\dfrac{1}{x^2} > 100$ if $|x| < \dfrac{1}{10}$

(b) $\dfrac{1}{|x-1|} > 1000$ if $|x-1| < \dfrac{1}{1000}$

(c) $\dfrac{-1}{(x-3)^2} < -1000$ if $|x-3| < \dfrac{1}{10\sqrt{10}}$

(d) $-\dfrac{1}{x^4} < -10000$ if $x^4 < \dfrac{1}{10000}$, $|x| < \dfrac{1}{10}$

54. **(a)** $\dfrac{1}{(x-1)^2} > 10$ if and only if $|x-1| < \dfrac{1}{\sqrt{10}}$

(b) $\dfrac{1}{(x-1)^2} > 1000$ if and only if $|x-1| < \dfrac{1}{10\sqrt{10}}$

(c) $\dfrac{1}{(x-1)^2} > 100000$ if and only if $|x-1| < \dfrac{1}{100\sqrt{10}}$

55. if $M > 0$ then $\dfrac{1}{(x-3)^2} > M$, $0 < (x-3)^2 < \dfrac{1}{M}$, $0 < |x-3| < \dfrac{1}{\sqrt{M}}$, $\delta = \dfrac{1}{\sqrt{M}}$

56. if $M<0$ then $\dfrac{-1}{(x-3)^2}<M$, $0<(x-3)^2<-\dfrac{1}{M}$, $0<|x-3|<\dfrac{1}{\sqrt{-M}}$, $\delta=\dfrac{1}{\sqrt{-M}}$

57. if $M>0$ then $\dfrac{1}{|x|}>M$, $0<|x|<\dfrac{1}{M}$, $\delta=\dfrac{1}{M}$

58. if $M>0$ then $\dfrac{1}{|x-1|}>M$, $0<|x-1|<\dfrac{1}{M}$, $\delta=\dfrac{1}{M}$

59. if $M<0$ then $-\dfrac{1}{x^4}<M$, $0<x^4<-\dfrac{1}{M}$, $|x|<\dfrac{1}{(-M)^{1/4}}$, $\delta=\dfrac{1}{(-M)^{1/4}}$

60. if $M>0$ then $\dfrac{1}{x^4}>M$, $0<x^4<\dfrac{1}{M}$, $x<\dfrac{1}{M^{1/4}}$, $\delta=\dfrac{1}{M^{1/4}}$

61. if $x>2$ then $|x+1-3|=|x-2|=x-2<\epsilon$ if $2<x<2+\epsilon$, $\delta=\epsilon$

62. if $x<1$ then $|3x+2-5|=|3x-3|=3|x-1|=3(1-x)<\epsilon$ if $1-x<\frac{1}{3}\epsilon$, $1-\frac{1}{3}\epsilon<x<1$, $\delta=\frac{1}{3}\epsilon$

63. if $x>4$ then $\sqrt{x-4}<\epsilon$ if $x-4<\epsilon^2$, $4<x<4+\epsilon^2$, $\delta=\epsilon^2$

64. if $x<0$ then $\sqrt{-x}<\epsilon$ if $-x<\epsilon^2$, $-\epsilon^2<x<0$, $\delta=\epsilon^2$

65. if $x>2$ then $|f(x)-2|=|x-2|=x-2<\epsilon$ if $2<x<2+\epsilon$, $\delta=\epsilon$

66. if $x<2$ then $|f(x)-6|=|3x-6|=3|x-2|=3(2-x)<\epsilon$ if $2-x<\frac{1}{3}\epsilon$, $2-\frac{1}{3}\epsilon<x<2$, $\delta=\frac{1}{3}\epsilon$

67. **(a)** if $M<0$ and $x>1$ then $\dfrac{1}{1-x}<M$, $x-1<-\dfrac{1}{M}$, $1<x<1-\dfrac{1}{M}$, $\delta=-\dfrac{1}{M}$

(b) if $M>0$ and $x<1$ then $\dfrac{1}{1-x}>M$, $1-x<\dfrac{1}{M}$, $1-\dfrac{1}{M}<x<1$, $\delta=\dfrac{1}{M}$

68. **(a)** if $M>0$ and $x>0$ then $\dfrac{1}{x}>M$, $x<\dfrac{1}{M}$, $0<x<\dfrac{1}{M}$, $\delta=\dfrac{1}{M}$

(b) if $M<0$ and $x<0$ then $\dfrac{1}{x}<M$, $-x<-\dfrac{1}{M}$, $\dfrac{1}{M}<x<0$, $\delta=-\dfrac{1}{M}$

69. **(a)** Given any $M>0$ there corresponds $N>0$ such that if $x>N$ then $f(x)>M$, $x+1>M$, $x>M-1$, $N=M-1$.

(b) Given any $M<0$ there corresponds $N<0$ such that if $x<N$ then $f(x)<M$, $x+1<M$, $x<M-1$, $N=M-1$.

70. **(a)** Given any $M>0$ there corresponds $N>0$ such that if $x>N$ then $f(x)>M$, $x^2-3>M$, $x>\sqrt{M+3}$, $N=\sqrt{M+3}$.

(b) Given any $M<0$ there corresponds $N<0$ such that if $x<N$ then $f(x)<M$, $x^3+5<M$, $x<(M-5)^{1/3}$, $N=(M-5)^{1/3}$.

71. if $\delta\le 2$ then $|x-3|<2$, $-2<x-3<2$, $1<x<5$, and $|x^2-9|=|x+3||x-3|<8|x-3|<\epsilon$ if $|x-3|<\frac{1}{8}\epsilon$, $\delta=\min\left(2,\frac{1}{8}\epsilon\right)$

72. **(a)** We don't care about the value of f at $x=a$, because the limit is only concerned with values of x near a. The condition that f be defined for all x (except possibly $x=a$) is necessary, because if some points were excluded then the limit may not exist; for example, let $f(x)=x$ if $1/x$ is not an integer and $f(1/n)=6$. Then $\lim_{x\to 0} f(x)$ does not exist but it would if the points $1/n$ were excluded.

(b) when $x<0$ then $\sqrt{x}$ is not defined

(c) yes; if $\delta\le 0.01$ then $x>0$, so $\sqrt{x}$ is defined

EXERCISE SET 2.5

1. **(a)** no, $x = 2$ **(b)** no, $x = 2$ **(c)** no, $x = 2$ **(d)** yes
(e) yes **(f)** yes

2. **(a)** no, $x = 2$ **(b)** no, $x = 2$ **(c)** no, $x = 2$ **(d)** yes
(e) no, $x = 2$ **(f)** yes

3. **(a)** no, $x = 1, 3$ **(b)** yes **(c)** no, $x = 1$ **(d)** yes
(e) no, $x = 3$ **(f)** yes

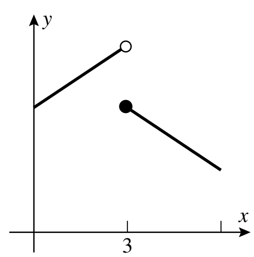
4. **(a)** no, $x = 3$ **(b)** yes **(c)** yes **(d)** yes
(e) no, $x = 3$ **(f)** yes

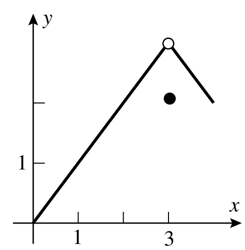
5. **(a)** At $x = 3$ the one-sided limits fail to exist.
(b) At $x = -2$ the two-sided limit exists but is not equal to $F(-2)$.
(c) At $x = 3$ the limit fails to exist.

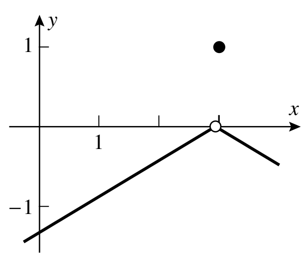
6. **(a)** At $x = 2$ the two-sided limit fails to exist.
(b) At $x = 3$ the two-sided limit exists but is not equal to $F(3)$.
(c) At $x = 0$ the two-sided limit fails to exist.

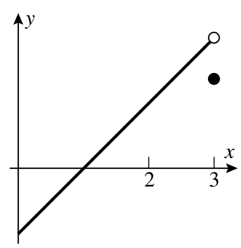
7. **(a)** 3 **(b)** 3

8. $-2/5$

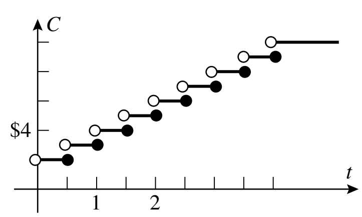
9. **(a)**


(b)


(c)


(d)


10. $f(x) = 1/x,\ g(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin \dfrac{1}{x} & \text{if } x \neq 0 \end{cases}$

11. **(a)**


(b) One second could cost you one dollar.

12. **(a)** no; disasters (war, flood, famine, pestilence, for example) can cause discontinuities
(b) continuous
(c) not usually continuous; see Exercise 11
(d) continuous

13. none

14. none

15. none

16. f is not defined at $x = \pm 1$

17. f is not defined at $x = \pm 4$

18. f is not defined at $x = \dfrac{-7 \pm \sqrt{57}}{2}$

19. f is not defined at $x = \pm 3$

20. f is not defined at $x = 0, -4$

21. none

22. f is not defined at $x = 0, -3$

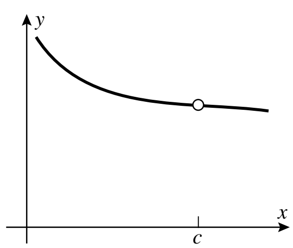
23. none; $f(x) = 2x + 3$ is continuous on $x < 4$ and $f(x) = 7 + \dfrac{16}{x}$ is continuous on $4 < x$; $\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^+} f(x) = f(4) = 11$ so f is continuous at $x = 4$

24. $\lim\limits_{x \to 1} f(x)$ does not exist so f is discontinuous at $x = 1$

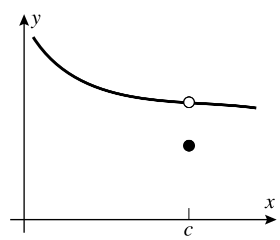
25. **(a)** f is continuous for $x < 1$, and for $x > 1$; $\lim\limits_{x \to 1^-} f(x) = 5$, $\lim\limits_{x \to 1^+} f(x) = k$, so if $k = 5$ then f is continuous for all x
(b) f is continuous for $x < 2$, and for $x > 2$; $\lim\limits_{x \to 2^-} f(x) = 4k$, $\lim\limits_{x \to 2^+} f(x) = 4 + k$, so if $4k = 4 + k$, $k = 4/3$ then f is continuous for all x

26. **(a)** no, f is not defined at $x = 2$
(b) no, f is not defined for $x \leq 2$
(c) yes
(d) no, f is not defined for $x \leq 2$

27. **(a)**

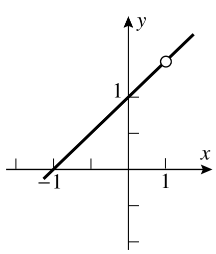


(b)

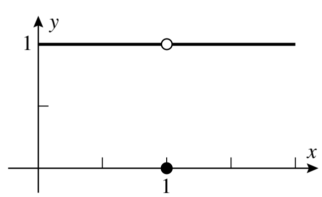


28. **(a)** $f(c) = \lim\limits_{x \to c} f(x)$

(b) $\lim\limits_{x \to 1} f(x) = 2$



$\lim\limits_{x \to 1} g(x) = 1$



(c) Define $f(1) = 2$ and redefine $g(1) = 1$.

29. **(a)** $x = 0$, $\lim_{x\to 0^-} f(x) = -1 \neq +1 = \lim_{x\to 0^+} f(x)$ so the discontinuity is not removable

(b) $x = -3$; define $f(-3) = -3 = \lim_{x\to -3} f(x)$, then the discontinuity is removable

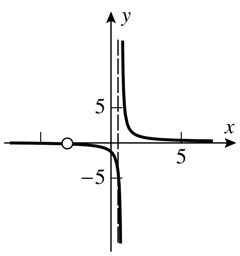
(c) f is undefined at $x = \pm 2$; at $x = 2$, $\lim_{x\to 2} f(x) = 1$, so define $f(2) = 1$ and f becomes continuous there; at $x = -2$, $\lim_{x\to -2}$ does not exist, so the discontinuity is not removable

30. **(a)** f is not defined at $x = 2$; $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{x+2}{x^2+2x+4} = \dfrac{1}{3}$, so define $f(2) = \dfrac{1}{3}$ and f becomes continuous there

(b) $\lim_{x\to 2^-} f(x) = 1 \neq 4 = \lim_{x\to 2^+} f(x)$, so f has a nonremovable discontinuity at $x = 2$

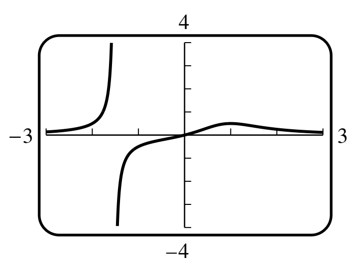
(c) $\lim_{x\to 1} f(x) = 8 \neq f(1)$, so f has a removable discontinuity at $x = 1$

31. **(a)** discontinuity at $x = 1/2$, not removable; at $x = -3$, removable



(b) $2x^2 + 5x - 3 = (2x-1)(x+3)$

32. **(a)** there appears to be one discontinuity near $x = -1.52$



(b) one discontinuity at $x = -1.52$

33. For $x > 0$, $f(x) = x^{3/5} = (x^3)^{1/5}$ is the composition (Theorem 2.4.6) of the two continuous functions $g(x) = x^3$ and $h(x) = x^{1/5}$ and is thus continuous. For $x < 0$, $f(x) = f(-x)$ which is the composition of the continuous functions $f(x)$ (for positive x) and the continuous function $y = -x$. Hence $f(-x)$ is continuous for all $x > 0$. At $x = 0$, $f(0) = \lim_{x\to 0} f(x) = 0$.

34. $x^4 + 7x^2 + 1 \geq 1 > 0$, thus $f(x)$ is the composition of the polynomial $x^4 + 7x^2 + 1$, the square root $\sqrt{x}$, and the function $1/x$ and is therefore continuous by Theorem 2.5.6.

35. **(a)** Let $f(x) = k$ for $x \neq c$ and $f(c) = 0$; $g(x) = l$ for $x \neq c$ and $g(c) = 0$. If $k = -l$ then $f + g$ is continuous; otherwise it's not.

(b) $f(x) = k$ for $x \neq c$, $f(c) = 1$; $g(x) = l \neq 0$ for $x \neq c$, $g(c) = 1$. If $kl = 1$, then fg is continuous; otherwise it's not.

36. A rational function is the quotient $f(x)/g(x)$ of two polynomials $f(x)$ and $g(x)$. By Theorem 2.5.2 f and g are continuous everywhere; by Theorem 2.5.3 f/g is continuous except when $g(x) = 0$.

37. Since f and g are continuous at $x = c$ we know that $\lim_{x \to c} f(x) = f(c)$ and $\lim_{x \to c} g(x) = g(c)$. In the following we use Theorem 2.2.2.

(a) $f(c) + g(c) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = \lim_{x \to c} (f(x) + g(x))$ so $f + g$ is continuous at $x = c$.

(b) same as (a) except the $+$ sign becomes a $-$ sign

(c) $\dfrac{f(c)}{g(c)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \lim_{x \to c} \dfrac{f(x)}{g(x)}$ so $\dfrac{f}{g}$ is continuous at $x = c$

38. $h(x) = f(x) - g(x)$ satisfies $h(a) > 0$, $h(b) < 0$. Use the Intermediate Value Theorem or Theorem 2.5.9.

39. Of course such a function must be discontinuous. Let $f(x) = 1$ on $0 \le x < 1$, and $f(x) = -1$ on $1 \le x \le 2$.

40. A square whose diagonal has length r has area $f(r) = r^2/2$. Note that $f(r) = r^2/2 < \pi r^2/2 < 2r^2 = f(2r)$. By the Intermediate Value Theorem there must be a value c between r and $2r$ such that $f(c) = \pi r^2/2$, i.e. a square of diagonal c whose area is $\pi r^2/2$.

41. The cone has volume $\pi r^2 h/3$. The function $V(r) = \pi r^2 h$ (for variable r and fixed h) gives the volume of a right circular cylinder of height h and radius r, and satisfies $V(0) < \pi r^2 h/3 < V(r)$. By the Intermediate Value Theorem there is a value c between 0 and r such that $V(c) = \pi r^2 h/3$, so the cylinder of radius c (and height h) has volume equal to that of the cone.

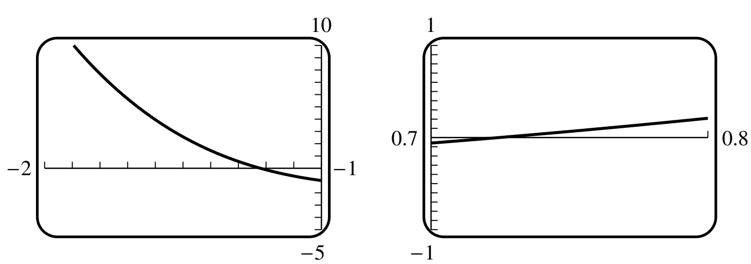
42. If $f(x) = x^3 - 4x + 1$ then $f(0) = 1$, $f(1) = -2$. Use Theorem 2.5.9.

43. If $f(x) = x^3 + x^2 - 2x$ then $f(-1) = 2$, $f(1) = 0$. Use the Intermediate Value Theorem.

44. Since $\lim_{x \to -\infty} p(x) = -\infty$ and $\lim_{x \to +\infty} p(x) = +\infty$ (or vice versa, if the leading coefficient of p is negative), it follows that for $M = -1$ there corresponds $N_1 < 0$, and for $M = 1$ there is $N_2 > 0$, such that $p(x) < -1$ for $x < N_1$ and $p(x) > 1$ for $x > N_2$. Choose $x_1 < N_1$ and $x_2 > N_2$ and use Theorem 2.5.9 on the interval $[x_1, x_2]$ to find a solution of $p(x) = 0$.

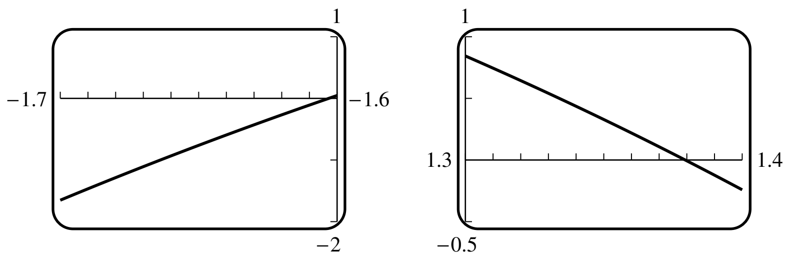
45. For the negative root, use intervals on the x-axis as follows: $[-2, -1]$; since $f(-1.3) < 0$ and $f(-1.2) > 0$, the midpoint $x = -1.25$ of $[-1.3, -1.2]$ is the required approximation of the root. For the positive root use the interval $[0, 1]$; since $f(0.7) < 0$ and $f(0.8) > 0$, the midpoint $x = 0.75$ of $[0.7, 0.8]$ is the required approximation.

46. $x = -1.25$ and $x = 0.75$.



47. For the negative root, use intervals on the x-axis as follows: $[-2, -1]$; since $f(-1.7) < 0$ and $f(-1.6) > 0$, use the interval $[-1.7, -1.6]$. Since $f(-1.61) < 0$ and $f(-1.60) > 0$ the midpoint $x = -1.605$ of $[-1.61, -1.60]$ is the required approximation of the root. For the positive root use the interval $[1, 2]$; since $f(1.3) > 0$ and $f(1.4) < 0$, use the interval $[1.3, 1.4]$. Since $f(1.37) > 0$ and $f(1.38) < 0$, the midpoint $x = 1.375$ of $[1.37, 1.38]$ is the required approximation.

48. $x = -1.605$ and $x = 1.375$.



49. $x = 2.24$

50. Set $f(x) = \dfrac{a}{x-1} + \dfrac{b}{x-3}$. Since $\lim\limits_{x\to 1^+} f(x) = +\infty$ and $\lim\limits_{x\to 3^-} f(x) = -\infty$ there exist $x_1 > 1$ and $x_2 < 3$ (with $x_2 > x_1$) such that $f(x) > 1$ for $1 < x < x_1$ and $f(x) < -1$ for $x_2 < x < 3$. Choose x_3 in $(1, x_1)$ and x_4 in $(x_2, 3)$ and apply Theorem 2.5.9 on $[x_3, x_4]$.

51. The uncoated sphere has volume $4\pi(x-1)^3/3$ and the coated sphere has volume $4\pi x^3/3$. If the volume of the uncoated sphere and of the coating itself are the same, then the coated sphere has twice the volume of the uncoated sphere. Thus $2(4\pi(x-1)^3/3) = 4\pi x^3/3$, or $x^3 - 6x^2 + 6x - 2 = 0$, with the solution $x = 4.847$ cm.

52. Let $g(t)$ denote the altitude of the monk at time t measured in hours from noon of day one, and let $f(t)$ denote the altitude of the monk at time t measured in hours from noon of day two. Then $g(0) < f(0)$ and $g(12) > f(12)$. Use Exercise 38.

53. We must show $\lim\limits_{x\to c} f(x) = f(c)$. Let $\epsilon > 0$; then there exists $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$. But this certainly satisfies Definition 2.4.1.

EXERCISE SET 2.6

1. none **2.** $x = \pi$ **3.** $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

4. $x = n\pi + \pi/2$, $n = 0, \pm 1, \pm 2, \ldots$ **5.** $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

6. none **7.** none **8.** $x = n\pi + \pi/2$, $n = 0, \pm 1, \pm 2, \ldots$

9. $2n\pi + \pi/6, 2n\pi + 5\pi/6$, $n = 0, \pm 1, \pm 2, \ldots$ **10.** none

11. **(a)** $\sin x, x^3 + 7x + 1$ **(b)** $|x|, \sin x$ **(c)** $x^3, \cos x, x + 1$
(d) $\sqrt{x}, 3 + x, \sin x, 2x$ **(e)** $\sin x, \sin x$ **(f)** $x^5 - 2x^3 + 1, \cos x$

12. **(a)** Use Theorem 2.5.6. **(b)** $g(x) = \cos x$, $g(x) = \dfrac{1}{x^2+1}$, $g(x) = x^2 + 1$

13. $\cos\left(\lim\limits_{x\to+\infty} \dfrac{1}{x}\right) = \cos 0 = 1$ **14.** $\sin\left(\lim\limits_{x\to+\infty} \dfrac{2}{x}\right) = \sin 0 = 0$

15. $\sin\left(\lim\limits_{x\to+\infty} \dfrac{\pi x}{2 - 3x}\right) = \sin\left(-\dfrac{\pi}{3}\right) = -\dfrac{\sqrt{3}}{2}$ **16.** $\dfrac{1}{2}\lim\limits_{h\to 0} \dfrac{\sin h}{h} = \dfrac{1}{2}$

17. $3\lim_{\theta\to 0}\dfrac{\sin 3\theta}{3\theta} = 3$

18. $\left(\lim_{\theta\to 0^+}\dfrac{1}{\theta}\right)\lim_{\theta\to 0^+}\dfrac{\sin\theta}{\theta} = +\infty$

19. $-\lim_{x\to 0^-}\dfrac{\sin x}{x} = -1$

20. $\dfrac{1}{3}\left(\lim_{x\to 0}\dfrac{\sin x}{x}\right)^2 = \dfrac{1}{3}$

21. $\dfrac{1}{5}\lim_{x\to 0^+}\sqrt{x}\lim_{x\to 0^+}\dfrac{\sin x}{x} = 0$

22. $\dfrac{\sin 6x}{\sin 8x} = \dfrac{6}{8}\dfrac{\sin 6x}{6x}\dfrac{8x}{\sin 8x}$, so $\lim_{x\to 0}\dfrac{\sin 6x}{\sin 8x} = \dfrac{6}{8} = \dfrac{3}{4}$

23. $\dfrac{\tan 7x}{\sin 3x} = \dfrac{7}{3\cos 7x}\dfrac{\sin 7x}{7x}\dfrac{3x}{\sin 3x}$ so $\lim_{x\to 0}\dfrac{\tan 7x}{\sin 3x} = \dfrac{7}{3(1)}(1)(1) = \dfrac{7}{3}$

24. $\left(\lim_{\theta\to 0}\sin\theta\right)\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 0$

25. $\left(\lim_{h\to 0}\cos h\right)\lim_{h\to 0}\dfrac{h}{\sin h} = 1$

26. $\dfrac{\sin h}{1-\cos h} = \dfrac{\sin h}{1-\cos h}\dfrac{1+\cos h}{1+\cos h} = \dfrac{\sin h(1+\cos h)}{1-\cos^2 h} = \dfrac{1+\cos h}{\sin h}$; no limit

27. $\dfrac{\theta^2}{1-\cos\theta}\dfrac{1+\cos\theta}{1+\cos\theta} = \dfrac{\theta^2(1+\cos\theta)}{1-\cos^2\theta} = \left(\dfrac{\theta}{\sin\theta}\right)^2(1+\cos\theta)$ so $\lim_{\theta\to 0}\dfrac{\theta^2}{1-\cos\theta} = (1)^2 2 = 2$

28. $\cos(\frac{1}{2}\pi - x) = \sin(\frac{1}{2}\pi)\sin x = \sin x$, so $\lim_{x\to 0}\dfrac{x}{\cos\left(\frac{1}{2}\pi - x\right)} = 1$

29. 0

30. $\dfrac{t^2}{1-\cos^2 t} = \left(\dfrac{t}{\sin t}\right)^2$, so $\lim_{t\to 0}\dfrac{t^2}{1-\cos^2 t} = 1$

31. $\dfrac{1-\cos 5h}{\cos 7h - 1} = \dfrac{(1-\cos 5h)(1+\cos 5h)(1+\cos 7h)}{(\cos 7h-1)(1+\cos 5h)(1+\cos 7h)} = -\dfrac{25}{49}\left(\dfrac{\sin 5h}{5h}\right)^2\left(\dfrac{7h}{\sin 7h}\right)^2\dfrac{1+\cos 7h}{1+\cos 5h}$ so $\lim_{h\to 0}\dfrac{1-\cos 5h}{\cos 7h - 1} = -\dfrac{25}{49}$

32. $\lim_{x\to 0^+}\sin\left(\dfrac{1}{x}\right) = \lim_{t\to +\infty}\sin t$; limit does not exist

33. $\lim_{x\to 0^+}\cos\left(\dfrac{1}{x}\right) = \lim_{t\to +\infty}\cos t$; limit does not exist

34. $\lim_{x\to 0}x - 3\lim_{x\to 0}\dfrac{\sin x}{x} = -3$

35. $2 + \lim_{x\to 0}\dfrac{\sin x}{x} = 3$

36.

5.1	5.01	5.001	5.0001	5.00001	4.9	4.99	4.999	4.9999	4.99999
0.098845	0.099898	0.99990	0.099999	0.100000	0.10084	0.10010	0.10001	0.10000	0.10000

The limit is 0.1.

37.

2.1	2.01	2.001	2.0001	2.00001	1.9	1.99	1.999	1.9999	1.99999
0.484559	0.498720	0.499875	0.499987	0.499999	0.509409	0.501220	0.500125	0.500012	0.500001

The limit is 0.5.

38.

−1.9	−1.99	−1.999	−1.9999	−1.99999	−2.1	−2.01	−2.001	−2.0001	−2.00001
−0.898785	−0.989984	−0.999000	−0.999900	−0.999990	−1.097783	−1.009983	−1.001000	−1.000100	−1.000010

The limit is −1.

39.

-0.9	-0.99	-0.999	-0.9999	-0.99999	-1.1	-1.01	-1.001	-1.0001	-1.00001
0.405086	0.340050	0.334001	0.333400	0.333340	0.271536	0.326717	0.332667	0.333267	0.333327

The limit is 1/3.

40. $k = f(0) = \lim_{x\to 0} \dfrac{\sin 3x}{x} = 3\lim_{x\to 0}\dfrac{\sin 3x}{3x} = 3$, so $k = 3$

41. $\lim_{x\to 0^-} f(x) = k \lim_{x\to 0} \dfrac{\sin kx}{kx\cos kx} = k$, $\lim_{x\to 0^+} f(x) = 2k^2$, so $k = 2k^2$, $k = \dfrac{1}{2}$

42. No; $\sin x/|x|$ has unequal one-sided limits.

43. **(a)** $\lim_{t\to 0^+}\dfrac{\sin t}{t} = 1$ **(b)** $\lim_{t\to 0^-}\dfrac{1-\cos t}{t} = 0$ (Theorem 2.6.3)

(c) $\sin(\pi - t) = \sin t$, so $\lim_{x\to\pi}\dfrac{\pi - x}{\sin x} = \lim_{t\to 0}\dfrac{t}{\sin t} = 1$

44. $\cos\left(\dfrac{\pi}{2} - t\right) = \sin t$, so $\lim_{x\to 2}\dfrac{\cos(\pi/x)}{x-2} = \lim_{t\to 0}\dfrac{(\pi - 2t)\sin t}{4t} = \lim_{t\to 0}\dfrac{\pi - 2t}{4}\lim_{t\to 0}\dfrac{\sin t}{t} = \dfrac{\pi}{4}$

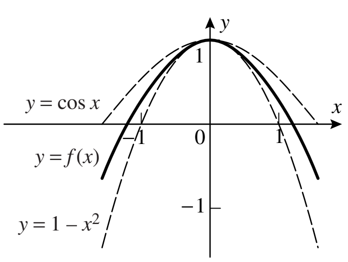
45. $t = x - 1$; $\sin(\pi x) = \sin(\pi t + \pi) = -\sin \pi t$; and $\lim_{x\to 1}\dfrac{\sin(\pi x)}{x-1} = -\lim_{t\to 0}\dfrac{\sin \pi t}{t} = -\pi$

46. $t = x - \pi/4$; $\tan x - 1 = \dfrac{2\sin t}{\cos t - \sin t}$; $\lim_{x\to\pi/4}\dfrac{\tan x - 1}{x - \pi/4} = \lim_{t\to 0}\dfrac{2\sin t}{t(\cos t - \sin t)} = 2$

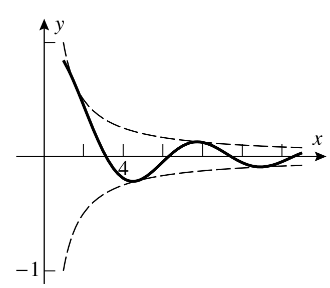
47. $-|x| \le x\cos\left(\dfrac{50\pi}{x}\right) \le |x|$

48. $-x^2 \le x^2\sin\left(\dfrac{50\pi}{\sqrt[3]{x}}\right) \le x^2$

49. $\lim_{x\to 0} f(x) = 1$ by the Squeezing Theorem

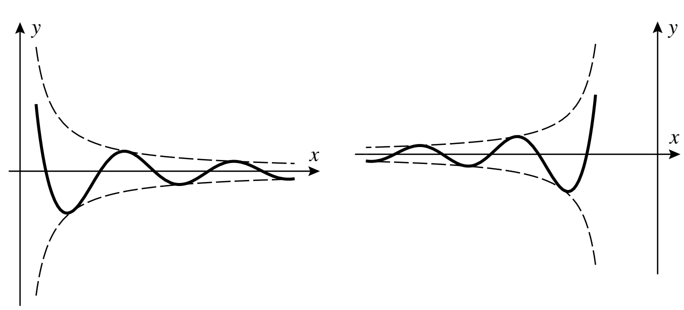


50. $\lim_{x\to+\infty} f(x) = 0$ by the Squeezing Theorem



51. Let $g(x) = -\dfrac{1}{x}$ and $h(x) = \dfrac{1}{x}$; thus $\lim_{x\to+\infty}\dfrac{\sin x}{x} = 0$ by the Squeezing Theorem.

52.



53. **(a)** $\sin x = \sin t$ where x is measured in degrees, t is measured in radians and $t = \dfrac{\pi x}{180}$. Thus $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{t\to 0} \dfrac{\sin t}{(180t/\pi)} = \dfrac{\pi}{180}$.

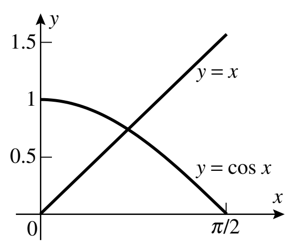
54. $\cos x = \cos t$ where x is measured in degrees, t in radians, and $t = \dfrac{\pi x}{180}$. Thus $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = \lim\limits_{t\to 0} \dfrac{1-\cos t}{(180t/\pi)} = 0$.

55. **(a)** $\sin 10^\circ = 0.17365$ **(b)** $\sin 10^\circ = \sin \dfrac{\pi}{18} \approx \dfrac{\pi}{18} = 0.17453$

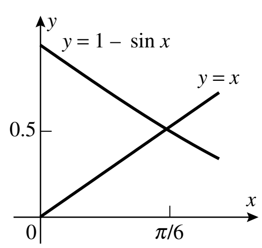
56. **(a)** $\cos\theta = \cos 2\alpha = 1 - 2\sin^2(\theta/2)$
$\approx 1 - 2(\theta/2)^2 = 1 - \frac{1}{2}\theta^2$
(b) $\cos 10^\circ = 0.98481$
(c) $\cos 10^\circ = 1 - \dfrac{1}{2}\left(\dfrac{\pi}{18}\right)^2 \approx 0.98477$

57. **(a)** 0.08749 **(b)** $\tan 5^\circ \approx \dfrac{\pi}{36} = 0.08727$

58. **(a)** $h = 52.55$ ft
(b) Since α is small, $\tan\alpha^\circ \approx \dfrac{\pi\alpha}{180}$ is a good approximation.
(c) $h \approx 52.36$ ft

59. **(a)** Let $f(x) = x - \cos x$; $f(0) = -1$, $f(\pi/2) = \pi/2$. By the IVT there must be a solution of $f(x) = 0$.
(b)


(c) 0.739

60. **(a)** $f(x) = x + \sin x - 1$; $f(0) = -1$, $f(\pi/6) = \pi/6 - 1/2 > 0$. By the IVT there must be a solution of $f(x) = 0$.
(b)


(c) $x = 0.511$

61. **(a)** There is symmetry about the equatorial plane.
(b) Let $g(\phi)$ be the given function. Then $g(38) < 9.8$ and $g(39) > 9.8$, so by the Intermediate Value Theorem there is a value c between 38 and 39 for which $g(c) = 9.8$ exactly.

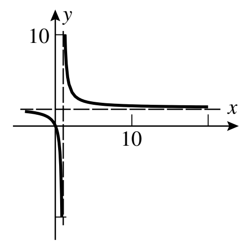
62. **(a)** does not exist **(b)** the limit is zero

(c) For part (a) consider the fact that given any $\delta > 0$ there are infinitely many rational numbers x satisfying $|x| < \delta$ and there are infinitely many irrational numbers satisfying the same condition. Thus if the limit were to exist, it could not be zero because of the rational numbers, and it could not be 1 because of the irrational numbers, and it could not be anything else because of *all* the numbers. Hence the limit cannot exist. For part (b) use the Squeezing Theorem with $+x$ and $-x$ as the 'squeezers'.

CHAPTER 2 SUPPLEMENTARY EXERCISES

1. **(a)** 1 **(b)** no limit **(c)** no limit
(d) 1 **(e)** 3 **(f)** 0
(g) 0 **(h)** 2 **(i)** 1/2

2. **(a)** $f(x) = 2x/(x-1)$ **(b)**



4. $f(x) = -1$ for $a \le x < \dfrac{a+b}{2}$ and $f(x) = 1$ for $\dfrac{a+b}{2} \le x \le b$

5. **(a)** $0.222\ldots, 0.24390, 0.24938, 0.24994, 0.24999, 0.25000$; for $x \neq 2$, $f(x) = \dfrac{1}{x+2}$, so the limit is $1/4$.

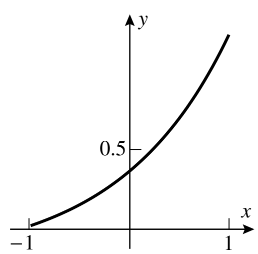
(b) $1.15782, 4.22793, 4.00213, 4.00002, 4.00000, 4.00000$; to prove, use $\dfrac{\tan 4x}{x} = \dfrac{\sin 4x}{x \cos 4x} = \dfrac{4}{\cos 4x}\dfrac{\sin 4x}{4x}$, the limit is 4.

6. **(a)** $y = 0$ **(b)** none **(c)** $y = 2$

7. **(a)**

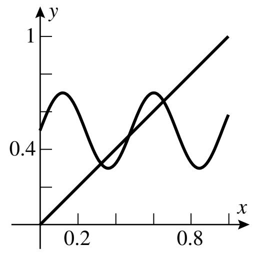
x	1	0.1	0.01	0.001	0.0001	0.00001	0.000001
$f(x)$	1.000	0.443	0.409	0.406	0.406	0.405	0.405

(b)



8. **(a)** 0.4 amperes **(b)** $[0.3947, 0.4054]$ **(c)** $\left[\dfrac{3}{7.5+\delta}, \dfrac{3}{7.5-\delta}\right]$

(d) 0.0187 **(e)** It becomes infinite.

9. **(a)** 

(b) Let $g(x) = x - f(x)$. Then $g(1) \geq 0$ and $g(0) \leq 0$; by the Intermediate Value Theorem there is a solution c in $[0, 1]$ of $g(c) = 0$.

10. **(a)** $\lim_{\theta \to 0} \tan\left(\frac{1-\cos\theta}{\theta}\right) = \tan\left(\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta}\right) = \tan\left(\lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\right) = \tan 0 = 0$

(b) $\frac{t-1}{\sqrt{t}-1} = \frac{t-1}{\sqrt{t}-1}\frac{\sqrt{t}+1}{\sqrt{t}+1} = \frac{(t-1)(\sqrt{t}+1)}{t-1} = \sqrt{t}+1$; $\lim_{t \to 1} \frac{t-1}{\sqrt{t}-1} = \lim_{t \to 1}(\sqrt{t}+1) = 2$

(c) $\frac{(2x-1)^5}{(3x^2+2x-7)(x^3-9x)} = \frac{(2-1/x)^5}{(3+2/x-7/x^2)(1-9/x^2)} \to 2^5/3 = 32/3$ as $x \to +\infty$

(d) $\sin(\theta+\pi) = \sin\theta\cos\pi - \cos\theta\sin\pi = -\sin\theta$, so $\lim_{\theta \to 0} \cos\left(\frac{\sin(\theta+\pi)}{2\theta}\right) = \lim_{\theta \to 0} \cos\left(\frac{-\sin\theta}{2\theta}\right)$
$= \cos\left(\lim_{\theta \to 0} \frac{-\sin\theta}{2\theta}\right) = \cos\left(-\frac{1}{2}\right)$

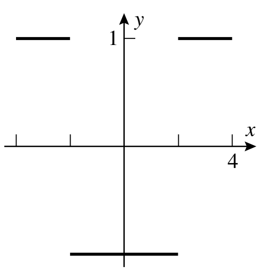
11. If, on the contrary, $f(x_0) < 0$ for some x_0 in $[0, 1]$, then by the Intermediate Value Theorem we would have a solution of $f(x) = 0$ in $[0, x_0]$, contrary to the hypothesis.

12. For $x < 2$ f is a polynomial and is continuous; for $x > 2$ f is a polynomial and is continuous. At $x = 2$, $f(2) = -13 \neq 13 = \lim_{x \to 2^+} f(x)$ so f is not continuous there.

13. $f(-6) = 185$, $f(0) = -1$, $f(2) = 65$; apply Theorem 2.4.9 twice, once on $[-6, 0]$ and once on $[0, 2]$

14. 3.317

15. Let $\epsilon = f(x_0)/2 > 0$; then there corresponds $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$, $-\epsilon < f(x) - f(x_0) < \epsilon$, $f(x) > f(x_0) - \epsilon = f(x_0)/2 > 0$ for $x_0 - \delta < x < x_0 + \delta$.

16. 

17. **(a)** $-3.449, 1.449$ **(b)** $x = 0, \pm 1.896$

18. Since $\lim_{x \to 0} \sin(1/x)$ does not exist, no conclusions can be drawn.

19. **(a)** $\sqrt{5}$, no limit, $\sqrt{10}$, $\sqrt{10}$, no limit, $+\infty$, no limit
(b) $5, 10, 0, 0, 10, -\infty, +\infty$

20. **(a)** $-1/5, +\infty, -1/10, -1/10$, no limit, 0, 0 **(b)** $-1, +1, -1, -1$, no limit, $-1, +1$

21. a/b **22.** 1 **23.** does not exist

24. 2 **25.** 0 **26.** k^2 **27.** $3-k$

28. The numerator satisfies: $|2x + x\sin 3x| \le |2x| + |x| = 3|x|$. Since the denominator grows like x^2, the limit is 0.

30. **(a)** $$\frac{\sqrt{x^2+4}-2}{x^2}\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} = \frac{x^2}{x^2(\sqrt{x^2+4}+2)} = \frac{1}{\sqrt{x^2+4}+2}, \text{ so}$$
$$\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2} = \frac{1}{4}$$

(b)

x	1	0.1	0.01	0.001	0.0001	0.00001
$f(x)$	0.236	0.2498	0.2500	0.2500	0.25000	0.00000

The division may entail division by zero (e.g. on an HP 42S), or the numerator may be inaccurate (catastrophic subtraction, e.g.).

(c) in the 3d picture, catastrophic subtraction

31.

x	0.1	0.01	0.001	0.0001	0.00001	0.000001
$f(x)$	2.59	2.70	2.717	2.718	2.7183	2.71828

32.

x	3.1	3.01	3.001	3.0001	3.00001	3.000001
$f(x)$	5.74	5.56	5.547	5.545	5.5452	5.54518

33.

x	1.1	1.01	1.001	1.0001	1.00001	1.000001
$f(x)$	0.49	0.54	0.540	0.5403	0.54030	0.54030

34.

x	0.1	0.01	0.001	0.0001	0.00001	0.000001
$f(x)$	99.0	9048.8	368063.3	4562.7	3.9×10^{-34}	0

35.

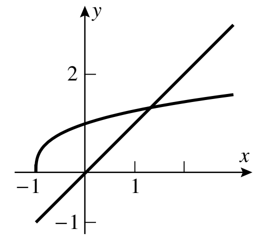
x	100	1000	10^4	10^5	10^6	10^7
$f(x)$	0.48809	0.49611	0.49876	0.49961	0.49988	0.49996

36. For large values of x (not much more than 100) the computer can't handle 5^x or 3^x, yet the limit is 5.

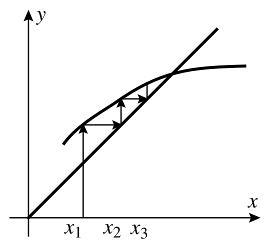
37. $\delta \approx 0.07747$ (use a graphing utility) **38.** \$2,001.60, \$2,009.66, \$2,013.62, \$2013.75

39. **(a)** $x^3 - x - 1 = 0,\ x^3 = x + 1,\ x = \sqrt[3]{x+1}.$

(b)

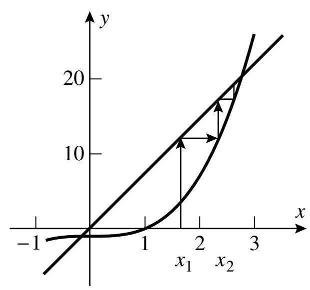


(c)



(d) $1, 1.26, 1.31, 1.322, 1.324, 1.3246, 1.3247$

40. **(a)**



(b) $0, -1, -2, -9, -730$

41. $x = \sqrt[5]{x+2}$; 1.267168

42. $x = \cos x$; 0.739085 (after 33 iterations!).